

# Projective Duality and Principal Nilpotent Elements of Symmetric Pairs

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*To A. L. Onishchik on the occasion of his 70th birthday.*

То, что не названо, — не существует.  
В. Набоков. Приглашение на казнь.

**ABSTRACT.** It is shown that projectivized irreducible components of nilpotent cones of complex symmetric spaces are projective self-dual algebraic varieties. Other properties equivalent to their projective self-duality are found.

**1.** Let  $\mathfrak{g}$  be a semisimple complex Lie algebra, let  $G$  be the adjoint group of  $\mathfrak{g}$ , and let  $\theta \in \text{Aut } \mathfrak{g}$  be an element of order 2. We set

$$\mathfrak{k} := \{x \in \mathfrak{g} \mid \theta(x) = x\}, \quad \mathfrak{p} := \{x \in \mathfrak{g} \mid \theta(x) = -x\}.$$

Then  $\mathfrak{k}$  and  $\mathfrak{p} \neq 0$ , the subalgebra  $\mathfrak{k}$  is reductive, and  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  is a  $\mathbb{Z}_2$ -grading of the Lie algebra  $\mathfrak{g}$ , cf., e.g., [OV]. Denote by  $G$  the adjoint group of  $\mathfrak{g}$ . The connected reductive algebraic subgroup  $K$  of  $G$  with the Lie algebra  $\mathfrak{k}$  is the adjoint group of  $\mathfrak{k}$ . Denoting the automorphism of  $G$  induced by  $\theta$  also by  $\theta$ , let  $K_\theta$  be the fixed point group of  $\theta$ . Then  $K$  is the identity component of  $K_\theta$ .

Let  $\mathcal{N}(\mathfrak{g})$  and  $\mathcal{N}(\mathfrak{p})$  be Zariski closed sets of all nilpotent elements in  $\mathfrak{g}$  and  $\mathfrak{p}$  respectively. They are cones (i.e., stable with respect to scalar multiplications and contain 0). We have

$$\mathcal{N}(\mathfrak{p}) = \mathcal{N}(\mathfrak{g}) \cap \mathfrak{p}.$$

The cone  $\mathcal{N}(\mathfrak{g})$  is irreducible, [K2], but  $\mathcal{N}(\mathfrak{p})$ , in general, is not, cf., e.g. [Se].

Consider the adjoint action of  $G$  on  $\mathfrak{g}$ . Then  $\mathfrak{p}$  and  $\mathcal{N}(\mathfrak{p})$  are  $K_\theta$ -stable. There are only finitely many  $G$ -orbits (resp.,  $K$ -orbits) in  $\mathcal{N}(\mathfrak{g})$  (resp.,  $\mathcal{N}(\mathfrak{p})$ ), [Dy], [K1], [KR]. Therefore  $\mathcal{N}(\mathfrak{g})$  (resp., every irreducible component of  $\mathcal{N}(\mathfrak{p})$ ) contains an open  $G$ -orbit  $\mathcal{N}(\mathfrak{g})_{\text{pr}}$  (resp.,  $K$ -orbit). Its elements are called *principal nilpotent elements* of  $\mathfrak{g}$  (resp.,  $\mathfrak{p}$ ). All principal nilpotent elements of  $\mathfrak{p}$  constitute a single

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1991 *Mathematics Subject Classification.* Primary 14L, 14M17, 17B70; Secondary 17B20, 14N05.

*Key words and phrases.* Semisimple Lie algebra, symmetric pair, nilpotent element, orbit, projective self-dual algebraic variety.

Partly supported by Russian grant HIII-123.2003.01 and by ETH Zürich, Switzerland.

$K_\theta$ -orbit  $\mathcal{N}(\mathfrak{p})_{\text{pr}}$ , [KR]. Hence the action of  $K_\theta$  on the set of irreducible components of  $\mathcal{N}(\mathfrak{p})$  is transitive. Remark that there are pairs  $(\mathfrak{g}, \theta)$  for which the intersection  $\mathcal{N}(\mathfrak{g})_{\text{pr}} \cap \mathfrak{p}$  is empty.

In the sequel, if  $V$  is a vector space and  $C$  is a cone in  $V$ , we denote by  $\mathbf{P}(V)$  the associated projective space of  $V$ , and by  $\mathbf{P}(C)$  the subset of  $\mathbf{P}(V)$  whose affine cone is  $C$ .

The Killing form  $\langle \cdot, \cdot \rangle$  of  $\mathfrak{g}$  (resp., its restriction  $\langle \cdot, \cdot \rangle|_{\mathfrak{p}}$  to  $\mathfrak{p}$ ) is nondegenerate and  $G$ -stable (resp.,  $K$ -stable). Hence we may (and shall) identify the linear spaces  $\mathfrak{g}$  and  $\mathfrak{g}^*$  (resp.,  $\mathfrak{p}$  and  $\mathfrak{p}^*$ ) by means of  $\langle \cdot, \cdot \rangle$  (resp.,  $\langle \cdot, \cdot \rangle|_{\mathfrak{p}}$ ). Then the projective dual  $\check{X}$  of any Zariski closed subset  $X$  of  $\mathbf{P}(\mathfrak{g}) = \mathbf{P}(\mathfrak{g}^*)$  (resp.,  $\mathbf{P}(\mathfrak{p}) = \mathbf{P}(\mathfrak{p}^*)$ ), cf., e.g., [Ha], becomes a Zariski closed subset of  $\mathbf{P}(\mathfrak{g})$  (resp.,  $\mathbf{P}(\mathfrak{p})$ ) as well. Given this, we call  $X$  *projective self-dual* if

$$X = \check{X}.$$

Now consider in  $\mathbf{P}(\mathfrak{g})$  and  $\mathbf{P}(\mathfrak{p})$  the Zariski closed subsets  $\mathbf{P}(\mathcal{N}(\mathfrak{g}))$  and  $\mathbf{P}(\mathcal{N}(\mathfrak{p}))$ . The projective dual of  $\mathbf{P}(\mathcal{N}(\mathfrak{g}))$  was identified in [P1]:

**THEOREM 1.** ([P1, Corollary 1 of Theorem 2]) *The variety  $\mathbf{P}(\mathcal{N}(\mathfrak{g}))$  is projective self-dual.*

The goal of this note is to prove the following

**THEOREM 2.** *Every irreducible component of  $\mathbf{P}(\mathcal{N}(\mathfrak{p}))$  is projective self-dual.*

We also prove several other theorems equivalent to Theorem 2.

Notice that the group  $K_\theta$  transitively permutes irreducible components of the variety  $\mathbf{P}(\mathcal{N}(\mathfrak{p}))$ . Hence they are isomorphic one another as embedded subvarieties of  $\mathbf{P}(\mathfrak{p})$ .

Some other nice geometric properties of the variety  $\mathbf{P}(\mathcal{N}(\mathfrak{p}))$  were discovered earlier. Namely, according to [KR],  $\mathbf{P}(\mathcal{N}(\mathfrak{p}))$  is a complete intersection in  $\mathbf{P}(\mathfrak{p})$  whose ideal is minimally generated by  $r$  homogeneous elements of the algebra  $\mathbb{C}[\mathfrak{p}]^K$  where  $r$  is the dimension of a Cartan subspace of  $\mathfrak{p}$  (i.e., a maximal linear subspace of  $\mathfrak{p}$  consisting of pairwise commuting semisimple elements). Their degrees are  $m_1 + 1, \dots, m_r + 1$ , where  $m_1, \dots, m_r$  are the exponents of the Weyl group  $W(\mathfrak{g}, \theta)$  of the symmetric pair  $(\mathfrak{g}, \theta)$ . The dimension and the degree of  $\mathbf{P}(\mathcal{N}(\mathfrak{p}))$  are equal respectively to  $\dim \mathfrak{p} - r - 1$  and the order of  $W(\mathfrak{g}, \theta)$ . In [B], [Sl], [Se], [O], [SS], for some symmetric pairs  $(\mathfrak{g}, \theta)$ , the generic singularities of  $\mathbf{P}(\mathcal{N}(\mathfrak{p}))$  were identified with some simplest rational singularities. For any  $(\mathfrak{g}, \theta)$ , every irreducible component of  $\mathbf{P}(\mathcal{N}(\mathfrak{p}))$  is the closure of some principal Hesselink stratum of  $\mathbf{P}(\mathcal{N}(\mathfrak{p}))$ , [He]. Therefore Hesselink's theory yields a desingularization of such component, cf. [PV], [P2] (see also [R]).

Theorem 1 follows from Theorem 2. Indeed, if  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$ , where  $\mathfrak{h}$  is a semisimple complex Lie algebra, and  $\theta((y, z)) = (z, y)$ , then

$$(1) \quad \mathfrak{k} = \{(y, y) \mid y \in \mathfrak{h}\}, \quad \mathfrak{p} = \{(y, -y) \mid y \in \mathfrak{h}\}.$$

If  $H$  is the adjoint group of  $\mathfrak{h}$ , then (1) implies that  $K$  is isomorphic to  $H$ , and the  $K$ -module  $\mathfrak{p}$  is isomorphic to the adjoint  $H$ -module  $\mathfrak{h}$ . Identifying these modules yields  $\mathcal{N}(\mathfrak{p}) = \mathcal{N}(\mathfrak{h})$ .

Finally remark that in the forthcoming paper [PT] we classified (listed) all  $K$ -orbits in  $\mathbf{P}(\mathcal{N}(\mathfrak{p}))$  such that their closures in  $\mathbf{P}(\mathfrak{p})$  are projective self-dual. The  $K$ -orbits in  $\mathbf{P}(\mathcal{N}(\mathfrak{p})_{\text{pr}})$  are not immediately identified in this classification. However

using some extra case by case arguments we can identify them and thereby obtain a proof of Theorem 2. The proof given in this note is different. It is short and free of case by case considerations.

**2.** Our approach to proving Theorem 2 is based on reducing it to an equivalent statement, Theorem 4, and then proving the latter. This reduction is based on the results of [P1].

To describe it, we introduce some notation. For any subset  $\mathfrak{x}$  of  $\mathfrak{g}$ , put

$$\mathfrak{x}^+ := \mathfrak{x} \cap \mathfrak{k}, \quad \mathfrak{x}^- := \mathfrak{x} \cap \mathfrak{p}.$$

If  $\mathfrak{x}$  and  $\mathfrak{y}$  are nonempty subsets of  $\mathfrak{g}$ , denote by  $\mathfrak{x}^\mathfrak{y}$  the centralizer of  $\mathfrak{y}$  in  $\mathfrak{x}$ ,

$$\mathfrak{x}^\mathfrak{y} := \{x \in \mathfrak{x} \mid [x, y] = 0 \text{ for all } y \in \mathfrak{y}\}.$$

If  $\mathfrak{x}$  is a linear subspace or a subalgebra of  $\mathfrak{g}$ , then  $\mathfrak{x}^\mathfrak{y}$  has this property as well. If  $\mathfrak{x}$  is  $\theta$ -stable and  $\mathfrak{y} \subseteq \mathfrak{k} \cup \mathfrak{p}$ , then  $\mathfrak{x}^\mathfrak{y}$  is  $\theta$ -stable.

**DEFINITION 1.** ([P1]) An element  $x \in \mathcal{N}(\mathfrak{p})$  and its  $K$ -orbit are called  $(-1)$ -distinguished if  $\mathfrak{p}^x$  contains no nonzero semisimple elements.

**REMARK 1.** This notion is a generalization of the notion of distinguished nilpotent element of a semisimple Lie algebra introduced in the Bala–Carter theory, [BC], cf., [CM], [M]. Indeed, in the notation of (1), an element of  $\mathcal{N}(\mathfrak{p})$  is  $(-1)$ -distinguished if and only if it is distinguished as the element of  $\mathcal{N}(\mathfrak{h})$ .

**THEOREM 3.** ([P1, Theorem 5]) *Let be  $x$  be a nonzero element of  $\mathcal{N}(\mathfrak{p})$  and let  $\overline{K \cdot x}$  be the closure of its  $K$ -orbit. Then the following properties are equivalent:*

- (i)  $\mathbf{P}(\overline{K \cdot x})$  is projective self-dual,
- (ii)  $x$  is  $(-1)$ -distinguished.

By Theorem 3, Theorem 2 is equivalent to the following

**THEOREM 4.** *Every principal nilpotent element of  $\mathcal{N}(\mathfrak{p})$  is  $(-1)$ -distinguished.*

**REMARK 2.** Since  $\mathcal{N}(\mathfrak{p})_{\text{pr}}$  is a single  $K_\theta$ -orbit, replacing ‘every’ with ‘some’ in Theorem 4 yields the equivalent statement. The same concerns Theorems 6, 7 and 8 below.

Given that Theorem 2 boils down to Theorem 4, below we key on proving Theorem 4.

In Subsections 6 and 7 we consider other interesting properties of principal nilpotent elements of  $\mathcal{N}(\mathfrak{p})$ . This yields other statements equivalent to Theorem 2.

**3.** Since reduction of Theorem 2 to Theorem 4 is crucial for our approach, first we sketch, for the sake of completeness, the proof of Theorem 3.

**PROOF.** The embedded tangent space to  $K \cdot x$  at  $x$  is  $[\mathfrak{k}, x]$ . For  $X := \mathbf{P}(\overline{K \cdot x})$ , the affine cone over  $\check{X}$  has the form  $\overline{K \cdot [\mathfrak{k}, x]^\perp}$ , where  $[\mathfrak{k}, x]^\perp$  is the orthogonal complement to  $[\mathfrak{k}, x]$  in  $\mathfrak{p}$  with respect to  $\langle \cdot, \cdot \rangle|_{\mathfrak{p}}$ . By the properties of the Killing form,  $[\mathfrak{k}, x]^\perp = \mathfrak{p}^x$ .

(i)  $\Rightarrow$  (ii): Assume that  $X = \check{X}$ . Then  $\mathfrak{p}^x \subseteq \overline{K \cdot \mathfrak{p}^x} = \overline{K \cdot x} \subseteq \mathcal{N}(\mathfrak{p})$ . Thus all nonzero elements of  $\mathfrak{p}^x$  are nilpotent, whence (ii).

(i)  $\Rightarrow$  (ii): Assume that  $x$  is  $(-1)$ -distinguished. Then, by the Jordan decomposition argument,  $\mathfrak{p}^x \subseteq \mathcal{N}(\mathfrak{p})$ , whence  $\check{X} \subseteq \mathbf{P}(\mathcal{N}(\mathfrak{p}))$ . Since there are only finitely

many  $K$ -orbits in  $\mathcal{N}(\mathfrak{p})$ , the last inclusion imples that the affine cone over  $\check{X}$  has the form  $\overline{K \cdot y}$  for some element  $y \in \mathcal{N}(\mathfrak{p})$ . From  $x \in \mathfrak{p}^x$  and  $y \in \mathfrak{p}^y$  we deduce that  $X \subseteq \check{X}$  and  $\check{X} \subseteq \check{\check{X}}$ . But  $\check{\check{X}} = X$  by the classical Biduality Theorem, cf. [Ha]. Whence (i).  $\square$

4. To prove Theorem 4 we need another condition equivalent to  $(-1)$ -distinguishness.

In the sequel, given a real or complex algebraic Lie algebra  $\mathfrak{h}$ , its reductive subalgebra  $\mathfrak{r}$  is called a *reductive Levi subalgebra* of  $\mathfrak{h}$  if  $\mathfrak{h}$  is a semidirect product of  $\mathfrak{r}$  and the unipotent radical  $\text{rad}_u \mathfrak{h}$  of  $\mathfrak{h}$ . The algebra  $\mathfrak{h}/\text{rad}_u \mathfrak{h}$  is called the *reductive Levi factor* of  $\mathfrak{h}$ .

LEMMA 1. *Let  $\mathfrak{h}$  be a  $\theta$ -stable algebraic subalgebra of  $\mathfrak{g}$  and let  $\mathfrak{r}$  be a  $\theta$ -stable reductive Levi subalgebra of  $\mathfrak{h}$ . Then the following properties are equivalent:*

- (i)  $\mathfrak{h}^-$  contains no nonzero semisimple elements,
- (ii)  $\mathfrak{r}^- = 0$ .

PROOF. Since  $\mathfrak{h}$  and  $\mathfrak{r}$  are  $\theta$ -stable, and  $\mathfrak{r}$  is a reductive Levi subalgebra of  $\mathfrak{h}$ , we have the following direct sum decompositions of vector spaces

$$(2) \quad \mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^-, \quad \mathfrak{r} = \mathfrak{r}^+ \oplus \mathfrak{r}^-, \quad \mathfrak{h} = \mathfrak{r} \oplus \text{rad}_u \mathfrak{h}.$$

Since  $\mathfrak{h}$  is  $\theta$ -stable,  $\text{rad}_u \mathfrak{h}$  is  $\theta$ -stable as well. Hence we have the decomposition

$$(3) \quad \text{rad}_u \mathfrak{h} = (\text{rad}_u \mathfrak{h})^+ \oplus (\text{rad}_u \mathfrak{h})^-.$$

(i)  $\Rightarrow$  (ii): Assume that (i) holds. If  $\mathfrak{r}^- \neq 0$ , then (2) implies that  $\theta|_{\mathfrak{r}} \in \text{Aut } \mathfrak{r}$  is an element of order 2. Hence, by [V], there is a nonzero  $\theta$ -stable algebraic torus in  $\mathfrak{r}^-$ . This contradicts (i). Whence  $\mathfrak{r}^- = 0$ .

(ii)  $\Rightarrow$  (i): Assume that (ii) holds. Then (2) implies that

$$(4) \quad \mathfrak{r} = \mathfrak{r}^+.$$

Plugging (4) and (3) in the last decomposition in (2), we deduce from the first decomposition in (2) that

$$(5) \quad \mathfrak{h}^- = (\text{rad}_u \mathfrak{h})^-.$$

Since all elements of  $\text{rad}_u \mathfrak{h}$  are nilpotent, (5) implies (i).  $\square$

Now let  $e$  be a nonzero element of  $\mathcal{N}(\mathfrak{p})$ . By Morozov's theorem,  $e$  can be embedded in an  $\mathfrak{sl}_2$ -triple  $\{e, h, f\}$ , i.e., an ordered triple of elements of  $\mathfrak{g}$  satisfying the bracket relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

By [KR, Proposition 4], we may (and shall) assume that

$$(6) \quad h \in \mathfrak{k}, \quad e, f \in \mathfrak{p}.$$

The linear span  $\mathfrak{s}$  of  $\{e, h, f\}$  is a three dimensional subalgebra of  $\mathfrak{g}$  isomorphic to  $\mathfrak{sl}_2$ . It is well known (cf., e.g., [CM, Lemma 3.7.3]) that  $\mathfrak{g}^{\mathfrak{s}}$  is a reductive Levi subalgebra of  $\mathfrak{g}^e$ . By (6), the subalgebras  $\mathfrak{g}^e$ ,  $\mathfrak{s}$  and  $\mathfrak{g}^{\mathfrak{s}}$  of  $\mathfrak{g}$  are  $\theta$ -stable. Hence Lemma 1 yields

LEMMA 2. *The following properties are equivalent:*

- (i)  $e$  is  $(-1)$ -distinguished,
- (ii)  $\mathfrak{p}^{\mathfrak{s}} = 0$ .

5. Now we can prove Theorem 4.

PROOF. According to the classical theory, we may (and shall) fix a  $\theta$ -stable real form  $\mathfrak{g}_{\mathbb{R}}$  of  $\mathfrak{g}$  such that

$$(7) \quad \mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}, \quad \text{where } \mathfrak{k}_{\mathbb{R}} := \mathfrak{g}_{\mathbb{R}}^+, \mathfrak{p}_{\mathbb{R}} := \mathfrak{g}_{\mathbb{R}}^-,$$

is a Cartan decomposition of  $\mathfrak{g}_{\mathbb{R}}$ , cf., e.g., [OV]. Let  $\mathfrak{a}_{\mathbb{R}}$  be a maximal abelian subspace of  $\mathfrak{p}_{\mathbb{R}}$  and let  $\mathfrak{a} \subset \mathfrak{p}$  be its complexification. Then  $\mathfrak{a}$  is a Cartan subspace of  $\mathfrak{p}$ , [KR, Lemma 2]. Consider the (restricted) root system  $\Delta$  of the pair  $(\mathfrak{g}, \mathfrak{a})$ . Fix in  $\mathfrak{a}_{\mathbb{R}}$  a closed Weyl chamber  $C$  of  $\Delta$  and let  $\Pi$  be the system of simple roots of  $\Delta$  corresponding to  $C$ .

Let  $e$  be a nonzero element of  $\mathcal{N}(\mathfrak{p})$ . Include it in an  $\mathfrak{sl}_2$ -triple  $\{e, h, f\}$  such that (6) holds, and let  $\mathfrak{s}$  be the linear span of  $\{e, h, f\}$ . By [KR, Lemma 6], replacing  $\{e, h, f\}$  with  $k \cdot \{e, h, f\}$  for an appropriate element  $k \in K$ , we may (and shall) assume that

$$(8) \quad c := e + f \in C.$$

By (8), the element  $c$  lies in  $\mathfrak{s}$ . Therefore

$$(9) \quad \mathfrak{k}^{\mathfrak{s}} \subseteq \mathfrak{k}^c, \quad \mathfrak{p}^{\mathfrak{s}} \subseteq \mathfrak{p}^c.$$

According to [KR, Proposition 13], the inclusion  $e \in \mathcal{N}(\mathfrak{p})_{\text{pr}}$  is equivalent to the property

$$(10) \quad \alpha(c) = 2 \quad \text{for all } \alpha \in \Pi.$$

Now assume that  $e$  lies in  $\mathcal{N}(\mathfrak{p})_{\text{pr}}$ . Then (10) implies that

$$(11) \quad \mathfrak{k}^c = \mathfrak{k}^{\mathfrak{a}}, \quad \mathfrak{p}^c = \mathfrak{a}.$$

Combining (9) and (11), we deduce from Lemma 2 that proving the statement of Theorem 4 is equivalent to proving the equality

$$(12) \quad \mathfrak{a}^{\mathfrak{s}} = 0.$$

Arguing on the contrary, assume that (12) does not hold. This means that there is a nonzero element  $z \in \mathfrak{a}$  commuting with every element of  $\mathfrak{s}$ . Consider the subalgebra  $\tilde{\mathfrak{g}}$  of  $\mathfrak{g}$  generated by  $\mathfrak{s}$  and  $\mathfrak{a}$ . Since  $\mathfrak{a}$  is commutative, the element  $z$  commutes with every element of  $\mathfrak{a}$  as well. Therefore it commutes with every element of the subalgebra  $\tilde{\mathfrak{g}}$ , i.e., belongs to its center. On the other hand, by [KR, Proposition 23], the algebra  $\tilde{\mathfrak{g}}$  is semisimple, and hence its center is 0. This contradiction completes the proof of Theorem 2.  $\square$

REMARK 3. It is well known that if  $x \in \mathcal{N}(\mathfrak{g})_{\text{pr}}$ , then every element of  $\mathfrak{g}^x$  is nilpotent, cf. [CM]. If  $x \in \mathcal{N}(\mathfrak{p})_{\text{pr}}$ , then  $\mathfrak{k}^x$ , in general, does not have this property. For instance, if  $\mathfrak{g}_{\mathbb{R}} = F_{4(-20)}$ , then the reductive Levi factor of  $\mathfrak{k}^x$  is  $G_2$ , see [Do, Table VIII].

6. The decomposition (7) defines an  $\mathbb{R}$ -structure on the algebraic group  $G$ . The identity component of the Lie group of  $\mathbb{R}$ -points of  $G$  is the adjoint group  $\text{Ad}(\mathfrak{g}_{\mathbb{R}})$  of  $\mathfrak{g}_{\mathbb{R}}$ . We set

$$\mathcal{N}(\mathfrak{g}_{\mathbb{R}}) := \mathcal{N}(\mathfrak{g}) \cap \mathfrak{g}_{\mathbb{R}}.$$

DEFINITION 2. ([PT]) An element  $x \in \mathcal{N}(\mathfrak{g}_{\mathbb{R}})$  is called *compact* if the reductive Levi factor of the centralizer  $\mathfrak{g}_{\mathbb{R}}^x$  is a compact Lie algebra.

Recall that there is a special bijection between the sets of nonzero  $K$ -orbits in  $\mathcal{N}(\mathfrak{p})$  and nonzero  $\text{Ad}(\mathfrak{g}_{\mathbb{R}})$ -orbits in  $\mathcal{N}(\mathfrak{g}_{\mathbb{R}})$ , cf. [CM], [M]. Namely, let  $\sigma$  be the complex conjugation of  $\mathfrak{g}$  defined by  $\mathfrak{g}_{\mathbb{R}}$ , viz.,

$$\sigma(a + ib) = a - ib, \quad a, b \in \mathfrak{g}_{\mathbb{R}}.$$

An  $\mathfrak{sl}_2$ -triple  $\{e, h, f\}$  in  $\mathfrak{g}$  satisfying (6) is called a *complex Cayley triple* if  $\sigma(e) = -f$ . For such a triple, set

$$e' := i(-h + e + f)/2, \quad h' := e - f, \quad f' := -i(h + e + f)/2.$$

Then  $\{e', h', f'\}$  is an  $\mathfrak{sl}_2$ -triple in  $\mathfrak{g}_{\mathbb{R}}$  such that  $\theta(e') = f'$ . An  $\mathfrak{sl}_2$ -triple in  $\mathfrak{g}_{\mathbb{R}}$  satisfying the last property is called a *real Cayley triple*. The map  $\{e, h, f\} \mapsto \{e', h', f'\}$  is a bijection from the set of complex to the set of real Cayley triples. The triple  $\{e, h, f\}$  is called the *Cayley transform* of  $\{e', h', f'\}$ .

Now let  $\mathcal{O}$  be a nonzero  $K$ -orbit in  $\mathcal{N}(\mathfrak{p})$ . Then, by [KR], there is a complex Cayley triple  $\{e, h, f\}$  in  $\mathfrak{g}$  such that  $e \in \mathcal{O}$ . Let  $\{e', h', f'\}$  be the real Cayley triple in  $\mathfrak{g}_{\mathbb{R}}$  such that  $\{e, h, f\}$  is its Cayley transform. Let  $\mathcal{O}' = \text{Ad}(\mathfrak{g}_{\mathbb{R}}) \cdot e'$ . Then the map assigning  $\mathcal{O}'$  to  $\mathcal{O}$  is well defined and establishes a bijection, called the *Kostant–Sekiguchi bijection*, between the set of nonzero  $K$ -orbits in  $\mathcal{N}(\mathfrak{p})$  and the set of nonzero  $\text{Ad}(\mathfrak{g}_{\mathbb{R}})$ -orbits in  $\mathcal{N}(\mathfrak{g}_{\mathbb{R}})$ .

**THEOREM 5.** ([PT, Theorem 5]) *Let  $\mathcal{O}$  be a nonzero  $K$ -orbit in  $\mathcal{N}(\mathfrak{p})$  and let  $x$  be an element of the  $\text{Ad}(\mathfrak{g}_{\mathbb{R}})$ -orbit in  $\mathcal{N}(\mathfrak{g}_{\mathbb{R}})$  corresponding to  $\mathcal{O}$  via the Kostant–Sekiguchi bijection. Then the following properties are equivalent:*

- (i)  $\mathcal{O}$  is  $(-1)$ -distinguished.
- (ii)  $x$  is compact.

Hence by Theorems 5 and 4, Theorem 2 is equivalent to the following

**THEOREM 6.** *For every  $K$ -orbit  $\mathcal{O}$  in  $\mathcal{N}(\mathfrak{p})_{\text{pr}}$ , the elements of the  $\text{Ad}(\mathfrak{g}_{\mathbb{R}})$ -orbit in  $\mathcal{N}(\mathfrak{g}_{\mathbb{R}})$  corresponding to  $\mathcal{O}$  via the Kostant–Sekiguchi bijection, are compact.*

**7.** Let  $e$  be a nonzero element of  $\mathcal{N}(\mathfrak{p})$ . Fix an  $\mathfrak{sl}_2$ -triple  $\{e, h, f\}$  such that (6) holds, and let  $\mathfrak{s}$  be the linear span of  $\{e, h, f\}$ . It is well known, cf., e.g., [CM], [M], that

$$\mathfrak{g} = \bigoplus_{d \in \mathbb{Z}} \mathfrak{g}_d, \quad \text{where } \mathfrak{g}_d := \{x \in \mathfrak{g} \mid [h, x] = dx\},$$

and that

$$\mathfrak{q} := \bigoplus_{d \geq 0} \mathfrak{g}_d, \quad \mathfrak{l} := \mathfrak{g}_0, \quad \mathfrak{u} = \bigoplus_{d > 0} \mathfrak{g}_d$$

are respectively a parabolic subalgebra of  $\mathfrak{g}$  (the ‘Jacobson–Morozov parabolic subalgebra’ of  $e$  that is actually uniquely determined by  $e$  alone), a reductive Levi subalgebra of  $\mathfrak{q}$  and the unipotent radical of  $\mathfrak{q}$ . By (6), each  $\mathfrak{g}_d$  is  $\theta$ -stable.

Since  $\mathfrak{s}$  is  $\theta$ -stable, the  $\mathfrak{s}$ -module  $\mathfrak{g}$  is a direct sum of  $\theta$ -stable simple submodules. From this and the inclusion  $e \in \mathfrak{g}_2^-$ , we deduce by means of the known argument based on the elementary representation theory of  $\mathfrak{sl}_2$  (cf., e.g., [CM, Lemma 8.2.1]), that the condition  $\mathfrak{p}^{\mathfrak{s}} = 0$  is equivalent to the condition

$$(13) \quad \dim \mathfrak{g}_0^- = \dim \mathfrak{g}_2^+.$$

Using Lemma 2, this yields the following graded analogue of the known criterion of distinguishness from the Bala–Carter theory (cf., e.g., [CM, Lemma 8.2.1]):

**LEMMA 3.** *In the notation of this subsection, the following properties are equivalent:*

- (i)  $e$  is  $(-1)$ -distinguished,
- (ii) the equality (13) holds.

Hence Theorem 2 is equivalent to the following

**THEOREM 7.** *The equality (13) holds for every element  $e \in \mathcal{N}(\mathfrak{p})_{\text{pr}}$  satisfying (6).*

Now assume that the element  $e$  is even, i.e.,  $\mathfrak{g}_d = 0$  for every odd  $d$ . Then we have the following graded analogue of another known criterion of distinguishness from the Bala–Carter theory (cf., e.g., [CM, Lemma 8.2.6]):

**LEMMA 4.** *In the notation of this subsection, if  $e$  is even, then the following properties are equivalent:*

- (i)  $e$  is  $(-1)$ -distinguished,
- (ii)  $\dim \mathfrak{l}^- = \dim \mathfrak{u}^+ / [\mathfrak{u}, \mathfrak{u}]^+$ .

**PROOF.** This is deduced from Lemma 3 using the argument analogous to the one used in the non-graded situation, cf., e.g., [CM, Theorem 8.2.6].  $\square$

**REMARK 4.** It is well known (cf., e.g., [CM, Theorem 8.2.3]) that every distinguished element of  $\mathcal{N}(\mathfrak{g})$  is even. In contrast to this, there are  $(-1)$ -distinguished elements in  $\mathcal{N}(\mathfrak{p})$  that are not even, [PT].

Since, by [KR, Theorem 4], the elements of  $\mathcal{N}(\mathfrak{p})_{\text{pr}}$  are even, Lemma 4 yields that Theorem 2 is equivalent to the following

**THEOREM 8.** *The equality  $\dim \mathfrak{l}^- = \dim \mathfrak{u}^+ / [\mathfrak{u}, \mathfrak{u}]^+$  holds for every element  $e \in \mathcal{N}(\mathfrak{p})_{\text{pr}}$  satisfying (6).*

**8.** In [N], it was made an attempt to develop an analogue of the Bala–Carter theory for nilpotent orbits in real semisimple Lie algebras. The Kostant–Sekiguchi bijection reduces this to finding an analogue of the Bala–Carter theory for  $K$ -orbits in  $\mathcal{N}(\mathfrak{p})$ . The theory developed in [N] is based on the notion of *noticed* nilpotent element. In the notation of Subsection 4, it is an element  $e \in \mathcal{N}(\mathfrak{p})$  characterized by the property  $\mathfrak{k}^s = 0$ . Recall that in contrast to this, our  $(-1)$ -distinguished element is characterized by the property  $\mathfrak{p}^s = 0$  (see Lemma 2).

The theory developed in [N] does not have some features that might be expected from a natural analogue of the Bala–Carter theory. As Theorem 3 shows, the geometric counterpart of distinguishness of a nonzero element  $x \in \mathcal{N}(\mathfrak{g})$  in the Bala–Carter theory is projective self-duality of the variety  $\mathbf{P}(\overline{G \cdot x})$ , and in the contents of symmetric pairs  $(\mathfrak{g}, \theta)$ , the algebraic counterpart of projective self-duality of  $\mathbf{P}(\overline{K \cdot x})$  for  $x \in \mathcal{N}(\mathfrak{p})$  is that  $x$  is  $(-1)$ -distinguished, not that  $x$  is noticed.

We believe that the results of [P1], this note and [PT] provide an evidence that if a natural analogue of the Bala–Carter theory for symmetric pairs exists, the notion of  $(-1)$ -distinguished element should play a key role in it, analogous to that of distinguished element in the original Bala–Carter theory.

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